RYUTHP-96/1
Feb., 1996
Sep.(Rev.)

The Renormalization-Group Method Applied to Asymptotic Analysis of Vector Fields

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Abstract

The renormalization group method of Goldenfeld, Oono and their collaborators is applied to asymptotic analysis of vector fields. The method is formulated on the basis of the theory of envelopes, as was done for scalar fields. This formulation actually completes the discussion of the previous work for scalar equations. It is shown in a generic way that the method applied to equations with a bifurcation leads to the Landau-Stuart and the (time-dependent) Ginzburg-Landau equations. It is confirmed that this method is actually a powerful theory for the reduction of the dynamics as the reductive perturbation method is. Some examples for ordinar diferential equations, such as the forced Duffing, the Lotka-Volterra and the Lorenz equations, are worked out in this method: The time evolution of the solution of the Lotka-Volterra equation is explicitly given, while the center manifolds of the Lorenz equation are constructed in a simple way in the RG method.

1 Introduction

It is well known that the renormalization group (RG) equations [1] have a peculiar power to improve the global nature of functions obtained in the perturbation theory in quantum field theory (QFT)[2]: The RG equations may be interpreted as representing the fact that the physical quantities $\mathcal{O}(p, \alpha, \mu)$ should not depend on the renormalization point μ having any arbitrary value,

$$\frac{\partial \mathcal{O}(p, \alpha; \mu)}{\partial \mu} = 0. \tag{1.1}$$

Such a floating renormalization point was first introduced by Gell-Mann and Low in the celebrated paper[1].

It is Goldenfeld, Oono and their collaborators (to be abbreviated to GO) [3, 4] who first showed that the RG equation can be used for purely mathematical problems as to improving the global nature of the solutions of differential equations obtained in the perturbation theory. One might say, however, that their presentation of the method is rather heuristic, heavily relied on the RG prescription in QFT and statistical physics; it seems that they were not so eager to give a mathematical reasoning to the method so that their method may be understandable even for those who are not familiar with the RG.[1] In fact, the reason why the RG equations even in QFT "improve" naive perturbation had not been elucidated. One may say that when GO successfully applied the RG method to purely mathematical problems such as solving differential equations, it had shaped a clear problem to reveal the mathematical reason of the powefullness of the RG method, at least, a la Stuckelberg-Peterman and Gell-Mann-Low.

Quite recently, the present author has formulated the method and given the reasoning of GO's method on the basis of the classical theory of envelopes[5, 6]: It was demonstrated that owing to the very RG equation, the functions consturcted from the solutions in the perturbation theory certainly satisfies the differential equation in question uniformly up to the order with which local solutions around $t = t_0$ is constructed. It was also indicated in a generic way that the RG equation may be regarded as the envelope equation. In fact, if a family of curves $\{\mathrm{C}_\mu\}_\mu$ in the x-y plane is represented by $y = f(x;\mu)$, the function $g(x)$ representing the envelope E is given by eliminating the parameter μ from the equation

$$\frac{\partial f(x;\mu)}{\partial \mu} = 0. \tag{1.2}$$

One can readily recognize the similarity of the envelope equation Eq.(1.2) with the RG equation Eq.(1.1). In Ref.'s[5, 6], a simplified prescription of the RG method is also presented. For instance, the perturbative expansion is made with respect to a small parameter and independent functions[2], and the procedure of the "renormalization" has been shown unnecessary.

However, the work given in [5, 6] may be said to be incomplete in the following sense: To give the proof mentioned above, the scalar equation in question was converted to a system of *first order* equations, which describe a vetor field. But the theory of envelopes for vetor fields, i.e.,envelopes of trajectories, has not been presented in [5, 6]. The theory should have been formulated for vector equations to make the discussion self-contained and complete.

One of the purposes of the present paper is therefore to reformulate geometrically the RG method for vector equations, i.e., systems of ODE's and PDE's and to complete the

[1] In Appendix A, we give a brief account of the Goldenfeld et al's prescription.

[2] Such an asmptotic series is called *generalizes asymptotic series*. The author is gratefull to T. Hatusda for telling him this fact and making him recognize its significance.

discussion given in [5, 6].

Another drawback of the previous work is that a reasoning given to a procedure to setting $t_0 = t$ in the RG method[3] of Goldenfeld et al was not fully persuasive.[4] In this paper, we present a more convincing reasoning for the procedure.

Once the RG method is formulated for vector fields, the applicability of the RG method developed by Goldenfeld, Oono and their collaborators is found to be wider than one might have imagined: The RG method is applicable also to, say,n-dimensional vector equations that are not simply converted to a scalar equation of the n-th order; needless to say, it is not necessarily possible to convert a system of ordinary differential equations (or dynamical system) to a scalar equation of a high order with a simple structure, although the converse is always possible trivially. For partial differential equations, it is not always possible to convert a system to a scalar equation of a high order[7]. Moreover, interesting equations in science including physics and applied mathematics are often given as a system. Therefore, it is of interest and importance to show that the RG method can be extended and applied to vector equations. To demonstrate the powefulness of the method, we shall work out some specific examples of vector equations.

We shall emphasize that the RG method provides a general method for the reduction of the dynamics as the reductive perturbation method (abbreviated to RP method)[8] does. It should be mentioned that Chen, Goldenfeld and Oono[4] already indicated that it is a rule that the RG equation gives equations for slow motions which the RP method may also describe. In this paper, we shall confirm their observation in a most general setting for vector equations. Furthermore, one can show [9] that the natural extension of the RG method also applies to *difference* equations or maps, and an extended envelope equation leads to a reduction of the dynamics even for discrete maps. Thus one sees that the RG method is truly a most promising candidate for a general theory of the reduction of the dynamics, although actual computation is often tediuous in such a general and mechanical method.

This paper is organized as follows: In the next section, we desribe the theory of envelopes for curves (or trajectories) in parameter representation. In section 3, the way how to construct envelope surfaces is given when a family of surfaces in three-dimensional space are parametrized with two parameters. In section 4, we give the basic mathematical theorem for the RG method applied to vector fields. This section is partially a recapitulation of a part of Ref.[5], although some clarifications are made here. In section 5, some examples are examined in this method, such as the forced Duffing[10], the Lotka-Volterra[11] and the Lorenz[12, 10] equations. The Duffing equation is also an example of non-autonomous one, containing an external force. In section 6, we treat generic equations with a bifurcation; the Landau-Stuart[13] and the Ginzburg-Landau equation will be derived in the RG method. The final section is devoted to a brief summary and concluding remarks. In Appendix A, a critical review of the Goldenfeld et al's method

[3]See Appendix A.

[4]The author is grateful to Y. Oono for his criticism on this point.

is given. In Appendix B, the Duffing equation is solved as a scalar equation in the RG method.

2 Envelopes of trajectories

To give a geometrical meaning to the RG equation for systems, one needs to formulate a theory of envelopes of curves which are given in a parameter representation: For example, if the equation is for $\mathbf{u}(t) = {}^t(x(t), y(t))$, the solution forms a trajectory or curve in the x-y plane with t being a parameter. In this section, we give a brief account of the classical theory of envelopes for curves in the n-dimensional space, given in a parameter representation.

Let a family of curves $\{\mathrm{C}_\alpha\}_\alpha$ in an n-dimensional space be given by

$$\mathbf{X}(t;\alpha) = {}^t(X_1(t;\alpha), X_2(t;\alpha), ..., X_n(t;\alpha)), \tag{2.1}$$

where the point $(X_1, X_2, ...X_n)$ moves in the n-dimensional space when t is varied. Curves in the family is parametrized by α. We suppose that the family of curves $\{\mathrm{C}_\alpha\}_\alpha$ has the envelope E:

$$\mathbf{X}_E(t) = {}^t(X_{E1}(t;\alpha), X_{E2}(t;\alpha), ..., X_{En}(t;\alpha)). \tag{2.2}$$

The functions $\mathbf{X}_E(t)$ may be obtained from $\mathbf{X}(t;\alpha)$ as follows. If the contact point of C_α and E is given by $t = t_\alpha$, we have

$$\mathbf{X}(t_\alpha;\alpha) = \mathbf{X}_E(t_\alpha). \tag{2.3}$$

For each point in E, there exists a parameter $\alpha = \alpha(t)$: Thus the envelope function is given by

$$\mathbf{X}_E(t_\alpha) = \mathbf{X}(t_\alpha;\alpha(t_\alpha)). \tag{2.4}$$

Then the problem is to get the function $\alpha(t)$, which is achieved as follows. The condition that E and C_α has the common tangent line at $\mathbf{X}(t_\alpha;\alpha) = \mathbf{X}_E(t_\alpha)$ reads

$$\frac{d\mathbf{X}}{dt}\Big|_{t=t_\alpha} = \frac{d\mathbf{X}_E}{dt}\Big|_{t=t_\alpha}. \tag{2.5}$$

On the other hand, differentiating Eq.(2.4), one has

$$\frac{d\mathbf{X}_E}{dt}\Big|_{t=t_\alpha} = \frac{\partial \mathbf{X}}{\partial t}\Big|_{t=t_\alpha} + \frac{\partial \mathbf{X}}{\partial \alpha}\frac{d\alpha}{dt}\Big|_{t=t_\alpha}. \tag{2.6}$$

From the last two equations, we get

$$\frac{\partial \mathbf{X}}{\partial \alpha} = \mathbf{0}. \tag{2.7}$$

From this equation, the function $\alpha = \alpha(t)$ is obtained. This is of the same form as the RG equation. Thus one may call the envelope equation the RG/E equation, too. In the application of the envelope theory for constructing global solutions of differential equations, the parameter is the initial time t_0, i.e., $\alpha = t_0$. Actually, apart from t_0, we have unknown functions given as initial values in the applications. We use the above condition to determine the t_0 dependence of the initial values by imposing that $t_0 = t$. In section 4, we shall show that the resultant function obtained as the envelope of the local solutions in the perturbation theory becomes an approximate but uniformly valid solution.

3 Envelope Surfaces

This section is devoted to give the condition for constructing the envelope surface of a family of surfaces with two parameters in the three-dimensional space. The generalization to the n-dimensional case is straightforward.

Let $\{\mathrm{S}_{\tau_1\tau_2}\}_{\tau_1\tau_2}$ be a family of surfaces given by

$$F(\mathbf{r}; \tau_1, \tau_2) = 0, \tag{3.1}$$

and E the envelope surface of it given by

$$G(\mathbf{r}) = 0, \tag{3.2}$$

with $\mathbf{r} = (x, y, z)$.

The fact that E contacts with $\mathrm{S}_{\tau_1\tau_2}$ at (x, y, z) implies

$$G(\mathbf{r}) = F(\mathbf{r}; \tau_1(\mathbf{r}), \tau_2(\mathbf{r})) = 0. \tag{3.3}$$

Let $(\mathbf{r} + d\mathbf{r}, \tau_1 + d\tau_1, \tau_2 + d\tau_2)$ gives another point in E, then

$$G(\mathbf{r} + d\mathbf{r}) = F(\mathbf{r} + d\mathbf{r}; \tau_1 + d\tau_1, \tau_2 + d\tau_2) = 0. \tag{3.4}$$

Taking the difference of the two equations, we have

$$\nabla F \cdot d\mathbf{r} + \frac{\partial F}{\partial \tau_1} d\tau_1 + \frac{\partial F}{\partial \tau_2} d\tau_2 = 0. \tag{3.5}$$

On the other hand, the fact that E and $\mathrm{S}_{\tau_1\tau_2}$ have a common tangent plane at $\mathbf{r}$ implies that

$$\nabla F \cdot d\mathbf{r} = 0. \tag{3.6}$$

Combining the last two equations, one has

$$\frac{\partial F}{\partial \tau_1} d\tau_1 + \frac{\partial F}{\partial \tau_2} d\tau_2 = 0. \tag{3.7}$$

Since $d\tau_1$ and $d\tau_2$ may be varied independently, we have

$$\frac{\partial F}{\partial \tau_1} = 0, \qquad \frac{\partial F}{\partial \tau_2} = 0. \tag{3.8}$$

From these equations, we get τ_i as a function of $\mathbf{r}$; $\tau_i = \tau_i(\mathbf{r})$.

As an example, let

$$F(x, y, z; \tau_1, \tau_2) = \mathrm{e}^{-\tau_1 y}\{1 - y(x - \tau_1)\} + \mathrm{e}^{-\tau_2 x}\{1 - x(y - \tau_2)\} - z. \tag{3.9}$$

The conditions $\partial F/\partial \tau_1 = 0$ and $\partial F/\partial \tau_2 = 0$ give

$$\tau_1 = x, \quad \tau_2 = y, \tag{3.10}$$

respectively. Hence one finds that the envelope is given by

$$G(x, y, z) = F(x, y, z; \tau_1 = x, \tau_2 = y) = 2\mathrm{e}^{-xy} - z = 0, \tag{3.11}$$

or $z = 2\exp(-xy)$.

It is obvious that the discussion can be extended to higher dimensional cases. In Ref.[6], envelope surfaces were constructed in multi steps when the RG method was applied to PDE's. However, as has been shown in this section, the construction can be performed by single step.

4 The basis of the RG method for systems

4.1 ODE's

Let $\mathbf{X} = {}^t(X_1, X_2, \cdots, X_n)$ and $\mathbf{F}(\mathbf{X}, t; \epsilon) = {}^t(F_1(\mathbf{X}, t; \epsilon),\ F_2(\mathbf{X}, t; \epsilon), \cdots, F_n(\mathbf{X}, t; \epsilon))$, and $\mathbf{X}$ satisfy the equation

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}, t; \epsilon). \tag{4.1}$$

Let us try to have the perturbation solution of Eq.(4.1) around $t = t_0$ by expanding

$$\mathbf{X}(t; t_0) = \mathbf{X}_0(t; t_0) + \epsilon \mathbf{X}_1(t; t_0) + \epsilon^2 \mathbf{X}_2(t; t_0) \cdots. \tag{4.2}$$

We suppose that an approximate solution $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}(t; t_0, \mathbf{W}(t_0))$ to the equation up to $O(\epsilon^p)$ is obtained,

$$\frac{d\tilde{\mathbf{X}}(t; t_0, \mathbf{W}(t_0))}{dt} = \mathbf{F}(\tilde{\mathbf{X}}(t), t; \epsilon) + O(\epsilon^p), \tag{4.3}$$

where the n-dimensional vector $\mathbf{W}(t_0)$ denotes the initial values assigned at the initial time $t = t_0$. Here notice that t_0 is arbitrary.

Let us construct the envelope function $\mathbf{X}_E(t)$ of the family of trajectories given by the functions $\tilde{\mathbf{X}}(t; t_0, \mathbf{W}(t_0))$ with t_0 parameterizing the trajectories. The construction is performed as follows: First we impose the RG/E equation, which now reads

$$\frac{d\tilde{\mathbf{X}}}{dt_0} = \mathbf{0}. \tag{4.4}$$

Notice that $\tilde{\mathbf{X}}$ contains the unknown function $\mathbf{W}(t_0)$ of t_0.[5] In the usual theory of envelopes, as given in section 2, this equation gives t_0 as a function of t. However, since we are now constructing the perturbation solution that is as close as possible to the exact one around $t = t_0$, we demand that the RG/E equation should give the solution $t_0 = t$, i.e., the parameter should coincide with the point of tangency. It means that the RG/E equation should determine the n-components of the initial vector $\mathbf{W}(t_0)$ so that $t_0 = t$. In fact, Eq.(4.4) may give equations as many as n which are independent of each other.[6] Thus the envelope function is given by

$$\mathbf{X}_E(t) = \tilde{\mathbf{X}}(t; t, \mathbf{W}(t)). \tag{4.5}$$

Then the fundamental theorem for the RG method is the following:
Theorem: $\mathbf{X}_E(t)$ *satisfies the original equation uniformly up to* $O(\epsilon^p)$.

Proof The proof is already given in Eq.(3.21) of Ref.[5]. Here we recapitulate it for completeness. $\forall t_0$, owing to the RG/E equation one has

$$\begin{aligned} \left.\frac{d\mathbf{X}_E}{dt}\right|_{t=t_0} &= \left.\frac{d\tilde{\mathbf{X}}(t; t_0, \mathbf{W}(t_0))}{dt}\right|_{t=t_0} + \left.\frac{d\tilde{\mathbf{X}}(t; t_0, \mathbf{W}(t_0))}{dt_0}\right|_{t=t_0}, \\ &= \left.\frac{d\tilde{\mathbf{X}}(t; t_0, \mathbf{W}(t_0))}{dt}\right|_{t=t_0}, \\ &= \mathbf{F}(\mathbf{X}_E(t_0), t_0; \epsilon) + O(\epsilon^p), \end{aligned} \tag{4.6}$$

where Eq.(4.4) has been used in the last equality. This concludes the proof.

[5] This means that Eq.(4.4) is a total derivative w.r.t. t_0;

$$\frac{d\tilde{\mathbf{X}}}{dt_0} = \frac{\partial\tilde{\mathbf{X}}}{\partial t_0} + \frac{d\mathbf{W}}{dt_0} \cdot \frac{\partial\tilde{\mathbf{X}}}{\partial\mathbf{W}} = \mathbf{0}.$$

[6] In the applications given below, the equation is, however, reduced to a scalar equation.

4.2 PDE's

It is desirable to develop a general theory for systems of PDE's as has been done for ODE's. But such a general theorem is not available yet. Nevertheless it *is* known that the simple generalization of Eq. (4.4) to envelope surfaces works.

Let $\tilde{\mathbf{X}}(t, \mathbf{x} : t_0, \mathbf{x}_0; \mathbf{W}(t_0, \mathbf{x}_0))$ is an approximate solution given in the perturbation theory up to $O(\epsilon^p)$ of a system of PDE's with respect to t and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Here we have made explicit that the solution has an initial and boundary value $\mathbf{W}(t_0, \mathbf{x}_0)$ dependent on t_0 and $\mathbf{x}_0 = (x_{10}, x_{20}, \ldots, x_{n0})$. As has been shown in section 3, the RG/E equation now reads

$$\frac{d\tilde{\mathbf{X}}}{dt_0} = \mathbf{0}, \quad \frac{d\tilde{\mathbf{X}}}{dx_{i0}} = \mathbf{0}, \quad (i = 1, 2, \ldots, n). \tag{4.7}$$

Notice again that $\tilde{\mathbf{X}}$ contains the unknown function $\mathbf{W}(t_0, \mathbf{x}_0)$ dependent on t_0 and $\mathbf{x}_0$, hence the derivatives are total derivatives. As the generalization of the case for ODE's, we demand that the RG/E equation should be compatible with the condition that the coordinate of the point of tangency becomes the parameter of the family of the surfaces;i.e.,

$$t_0 = t, \quad \mathbf{x}_0 = \mathbf{x}. \tag{4.8}$$

Then the RG/E equation is now reduced to equations for the unknown function $\mathbf{W}$, which will be shown to be the amplitude equations such as time-dependent Ginzburg-Landau equation. Here we remark that although Eq.(4.7) is a vector equation, the equation to appear below will be reduced to a scalar one; see subsection 6.2. It can be shown, at least for equations treated so far and here, that the resultant envelope functions satisfy the original equations uniformly up to $O(\epsilon^p)$; see also Ref.[6].

5 Simple examples

In this section, we treat a few of simple examples of systems of ODE's to show the how the RG method works. The examples are the Duffing[10] equation of non-autonomous nature, the Lotka-Volterra[11] and the Lorenz[12] equation. The first one may be treated as a scalar equation. Actually, the equation is easier to calculate when treated as a scalar one. We give such a treatment in Appendix B. We shall work out to derive the time dependence of the solution to the Lotka-Volterra equation explicitly. The last one is an example with three degrees of freedom, which shows a bifurcation[10]. We shall give the center manifolds to this equation around the first bifurcation of the Lorenz model. A general treatment for equations with a bifurcation will be treated in section 6.

5.1 Forced Duffing equation

The forced Duffing equations are reduced to

$$\begin{aligned} \ddot{x} + 2\epsilon\gamma\dot{x} + (1+\epsilon\sigma)x + \epsilon h x^3 &= \epsilon f \cos t, \\ \ddot{y} + 2\epsilon\gamma\dot{y} + (1+\epsilon\sigma)y + \epsilon h y^3 &= \epsilon f \sin t. \end{aligned} \tag{5.1}$$

Defining a complex variable $z = x + iy$, one has

$$\ddot{z} + 2\epsilon\gamma\dot{z} + (1+\epsilon\sigma)z + \frac{\epsilon h}{2}(3|z|^2 z + z^{*3}) = \epsilon f \mathrm{e}^{it}. \tag{5.2}$$

We suppose that ϵ is small.

We convert the equation to the system

$$\left(\frac{d}{dt} - L_0\right)\mathbf{u} = -\epsilon F(\xi, \eta; t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{5.3}$$

where

$$\begin{aligned} \mathbf{u} &= \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad \xi = z, \quad \eta = \dot{z}, \\ L_0 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \end{aligned} \tag{5.4}$$

and

$$F(\xi, \eta; t) = \sigma\xi + 2\gamma\eta\frac{h}{2}(3|\xi|^2 + \xi^{*3}) - f\mathrm{e}^{it}. \tag{5.5}$$

Let us first solve the equation in the perturbation theory by expanding

$$\mathbf{u} = \mathbf{u}_0 + \epsilon\mathbf{u}_1 + \ldots, \tag{5.6}$$

with $\mathbf{u}_i = {}^t(\xi_i, \eta_i)$ $(i = 0, 1, \ldots)$. We only have to solve the following equations successively;

$$\begin{aligned} \left(\frac{d}{dt} - L_0\right)\mathbf{u}_0 &= \mathbf{0}, \\ \left(\frac{d}{dt} - L_0\right)\mathbf{u}_1 &= -F(\xi_0, \eta_0; t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned} \tag{5.7}$$

and so on. The solution of the zero-th order equation is found to be

$$\mathbf{u}_0(t; t_0) = W(t_0)\mathbf{U}\mathrm{e}^{it}, \tag{5.8}$$

where $\mathbf{U}$ is an eigenvector belonging to an eigen value i of L_0,

$$L_0\mathbf{U} = i\mathbf{U}, \quad \mathbf{U} = \begin{pmatrix} 1 \\ i \end{pmatrix}. \tag{5.9}$$

The other eigenvector is given by the complex conjugate $\mathbf{U}^*$, which belongs to the other eigenvalue $-i$. We have made it explicit that the constant W may be dependent on the initial time t_0. In terms of the component,

$$\xi_0(t;t_0) = W(t_0)\mathrm{e}^{it}, \quad \eta_0(t;t_0) = iW(t_0)\mathrm{e}^{it}. \tag{5.10}$$

Inserting these into $F(\xi_0, \eta_0; t)$, one has

$$F(\xi_0, \eta_0; t) = \mathcal{W}(t_0)\mathrm{e}^{it} + \frac{h}{2}W^{*3}\mathrm{e}^{-3it}, \tag{5.11}$$

with

$$\mathcal{W}(t_0) \equiv (\sigma + 2i\gamma)W + \frac{3h}{2}|W|^2W - f \tag{5.12}$$

We remark that the inhomogeneous term includes a term proportional to the zero-th order solution. Thus $\mathbf{u}_1$ contains a resonance or a secular term as follows;

$$\mathbf{u}_1(t;t_0) \quad = \quad -\frac{1}{2i}\mathcal{W}\mathrm{e}^{it}\{(t - t_0 + \frac{1}{2i})\mathbf{U} - \frac{1}{2i}\mathbf{U}^*\} - \frac{h}{16}W^{*3}\mathrm{e}^{-3it}(\mathbf{U} - 2\mathbf{U}^*). \tag{5.13}$$

In terms of the components

$$\begin{aligned} \xi_1(t;t_0) &= \frac{i}{2}\mathcal{W}\mathrm{e}^{it}(t - t_0) + \frac{h}{16}W^{*3}\mathrm{e}^{-3it}, \\ \eta_1(t;t_0) &= -\frac{\mathcal{W}}{2}\mathrm{e}^{it}(t - t_0 - i) - \frac{3i}{16}hW^{*3}\mathrm{e}^{-3it}. \end{aligned} \tag{5.14}$$

Adding the terms, we have

$$\begin{aligned} \mathbf{u}(t) &\simeq \mathbf{u}_0(t;t_0) + \epsilon\mathbf{u}_1(t;t_0), \\ &= W(t_0)\mathbf{U}\mathrm{e}^{it} - \epsilon\frac{1}{2i}\mathcal{W}\mathrm{e}^{it}\{(t - t_0 + \frac{1}{2i})\mathbf{U} - \frac{1}{2i}\mathbf{U}^*\} - \epsilon\frac{h}{16}W^{*3}\mathrm{e}^{-3it}(\mathbf{U} - 2\mathbf{U}^*), \\ &\equiv \tilde{\mathbf{u}}(t;t_0). \end{aligned} \tag{5.15}$$

In terms of the components,

$$\begin{aligned} \xi(t;t_0) &\simeq W(t_0)\mathrm{e}^{it} + \epsilon\frac{i}{2}\mathcal{W}(t_0)\mathrm{e}^{it}(t - t_0) + \epsilon\frac{h}{16}W^{*3}\mathrm{e}^{-3it} \equiv \tilde{\xi}, \\ \eta(t;t_0) &\simeq iW(t_0)\mathrm{e}^{it} - \epsilon\frac{\mathcal{W}}{2}\mathrm{e}^{it}(t - t_0 - i) - \epsilon\frac{3i}{16}hW^{*3}\mathrm{e}^{-3it} \equiv \tilde{\eta}. \end{aligned} \tag{5.16}$$

Now let us construct the envelope $\mathbf{u}_E(t)$ of the family of trajectories or curves $\tilde{\mathbf{u}}(t;t_0) = (\tilde{\xi}(t;t_0), \tilde{\eta}(t;t_0))$ which is parametrized with t_0; $\mathbf{u}_E(t)$ will be found to be an approximate solution to Eq. (5.3) in the global domain. According to section 2, the envelope may be obtained from the equation

$$\frac{d\tilde{\mathbf{u}}(t;t_0)}{dt_0} = 0. \tag{5.17}$$

In the usual procedure for constructing the envelopes, the above equation is used for obtaining t_0 as a function of t, and the resulting $t_0 = t_0(t)$ is inserted in $\tilde{\mathbf{u}}(t;t_0)$ to make the envelope function $\mathbf{u}_E(t) = \tilde{\mathbf{u}}(t;t_0(t))$. In our case, we are constructing the envelope around $t = t_0$, so we rather impose that

$$t_0 = t, \tag{5.18}$$

and Eq.(5.17) is used to obtain the initial value $W(t_0)$ as a function of t_0. That is, we have

$$\begin{aligned} 0 &= \left.\frac{d\tilde{\mathbf{u}}(t;t_0)}{dt_0}\right|_{t_0=t}, \\ &= \frac{dW}{dt}\mathbf{U}\mathrm{e}^{it} + \epsilon\frac{\mathcal{W}}{2i}\mathrm{e}^{it}\mathbf{U} + \epsilon\frac{i}{2}\frac{d\mathcal{W}}{dt}\mathrm{e}^{it}\frac{1}{2i}(\mathbf{U} - \mathbf{U}^*) - \frac{3\epsilon h}{16}\frac{dW^*}{dt}\mathrm{e}^{-3it}(\mathbf{U} - 2\mathbf{U}^*). \end{aligned} \tag{5.19}$$

Noting that the equation is consistent with $dW/dt = O(\epsilon)$, one has

$$\begin{aligned} \frac{dW}{dt} &= i\frac{\epsilon}{2}\mathcal{W}(t), \\ &= i\frac{\epsilon}{2}\{(\sigma + 2i\gamma)W(t) + \frac{3h}{2}|W(t)|^2 W(t) - f\}. \end{aligned} \tag{5.20}$$

This is the amplitude equation called Landau-Stuart equation, which may be also given by the RP method[8] as a reduction of the dynamics. With this equation, the envelope trajectory is given by

$$\begin{aligned} \xi_E(t) &= W(t)\mathrm{e}^{it} + \epsilon\frac{h}{16}W^{*3}\mathrm{e}^{-3it}, \\ \eta_E(t) &= i(W(t) + \epsilon\frac{1}{2}\mathcal{W}(t))\mathrm{e}^{it} - \epsilon\frac{3i}{16}hW^{*3}\mathrm{e}^{-3it}. \end{aligned} \tag{5.21}$$

For completeness, let us examine the stationary solution of the Landau-Stuart equation, briefly;

$$\mathcal{W} = (\sigma + 2i\gamma)W + \frac{3}{2}\epsilon h|W|^2 W - f = 0. \tag{5.22}$$

Writing W as

$$W = A\mathrm{e}^{-i\theta}, \tag{5.23}$$

we have

$$A^2\Big[(\frac{3}{2}hA^2 + \sigma)^2 + 4\gamma^2\Big] = f^2, \tag{5.24}$$

which describes the jumping phenomena of the Duffing oscillator.

5.2 Lotka-Volterra equation

As another simple example, we take the Lotka-Volterra equation[11];

$$\dot{x} = ax - \epsilon xy, \qquad \dot{y} = -by + \epsilon' xy, \tag{5.25}$$

where the constants a, b, ϵ and ϵ' are assumed to be positive. It is well known that the equation has the conserved quantity, i.e.,

$$b \ln |x| + a \ln |y| - (\epsilon' x + \epsilon y) = \text{const.}. \tag{5.26}$$

The fixed points are given by $(x = 0, y = 0)$ and $(x = b/\epsilon', y = a/\epsilon)$. Shifting and scaling the variables by

$$x = (b + \epsilon\xi)/\epsilon', \qquad y = a/\epsilon + \eta, \tag{5.27}$$

we get the reduced equation given by the system

$$\left(\frac{d}{dt} - L_0\right)\mathbf{u} = -\epsilon\xi\eta \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{5.28}$$

where

$$\mathbf{u} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \qquad L_0 = \begin{pmatrix} 0 & -b \\ a & 0 \end{pmatrix}. \tag{5.29}$$

The eigen value equation

$$L_0 \mathbf{U} = \lambda_0 \mathbf{U} \tag{5.30}$$

has the solution

$$\lambda_0 = \pm i\sqrt{ab} \equiv \pm i\omega, \qquad \mathbf{U} = \begin{pmatrix} 1 \\ \mp i\frac{\omega}{b} \end{pmatrix}. \tag{5.31}$$

Let us try to apply the perturbation theory to solve the equation by expanding the variable in a Taylor series of ϵ;

$$\mathbf{u} = \mathbf{u}_0 + \epsilon\mathbf{u}_1 + \epsilon^2\mathbf{u}_2 + \cdots, \tag{5.32}$$

with $\mathbf{u}_i = {}^t(\xi_i, \eta_i)$. The lowest term satisfies the equation

$$\left(\frac{d}{dt} - L_0\right)\mathbf{u}_0 = \mathbf{0}, \tag{5.33}$$

which yields the solution

$$\mathbf{u}_0(t; t_0) = W(t_0)\mathrm{e}^{i\omega t}\mathbf{U} + \text{c.c.}, \tag{5.34}$$

or

$$\xi_0 = W(t_0)\mathrm{e}^{i\omega t} + \text{c.c.}, \qquad \eta_0 = -\frac{\omega}{b}(iW(t_0)\mathrm{e}^{i\omega t} + \text{c.c.}). \tag{5.35}$$

Here we have supposed that the initial value W depends on the initial time t_0.

Noting that

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \alpha \mathbf{U} + \text{c.c.}, \tag{5.36}$$

with $\alpha = (1 - ib/\omega)/2$, one finds that the first order term satisfies the equation

$$\Big(\frac{d}{dt} - L_0\Big)\mathbf{u}_1 = \frac{\omega}{b}\Big[iW^2\mathrm{e}^{2i\omega t}(\alpha\mathbf{U} + \text{c.c.}) + \text{c.c.}\Big], \tag{5.37}$$

the solution to which is found to be

$$\mathbf{u}_1 = \frac{1}{b}\Big[W^2(\alpha\mathbf{U} + \frac{\alpha^*}{3}\mathbf{U}^*)\mathrm{e}^{2i\omega t} + \text{c.c.}\Big], \tag{5.38}$$

or

$$\begin{aligned} \xi_1 &= \frac{1}{b}(\frac{2\omega - ib}{3\omega}W^2\mathrm{e}^{2i\omega t} + \text{c.c.}), \\ \eta_1 &= -\frac{\omega}{3b^2}(\frac{2b + i\omega}{\omega}W^2\mathrm{e}^{2i\omega t} + \text{c.c.}). \end{aligned} \tag{5.39}$$

The second order equation now reads

$$\Big(\frac{d}{dt} - L_0\Big)\mathbf{u}_2 = \frac{1}{3b^2}\Big[\{(b - i\omega)|W|^2W\mathrm{e}^{i\omega t} + 3(b + i\omega)W^3\mathrm{e}^{3i\omega t}\} + \text{c.c.}\Big]\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{5.40}$$

We remark that the inhomogeneous term has a part proportional to the zero-th-order solution, which gives rise to a resonance. Hence the solution necessarily includes secular terms as follows;

$$\begin{aligned} \mathbf{u}_2 \;=\; & \Big[\frac{b - i\omega}{3b^2}|W|^2W\Big\{\alpha(t - t_0 + i\frac{\alpha^*}{2\omega})\mathbf{U} + \frac{\alpha^*}{2i\omega}\mathbf{U}^*\Big\}\mathrm{e}^{i\omega t} \\ & + \frac{b + i\omega}{4b^2 i\omega}W^3(2\alpha\mathbf{U} + \alpha^*\mathbf{U}^*)\mathrm{e}^{3i\omega t}\Big] + \text{c.c.}. \end{aligned} \tag{5.41}$$

In terms of the components, one finds

$$\begin{aligned} \xi_2 \;=\; & \Big[\frac{-i}{6\omega}\frac{b^2 + \omega^2}{b^2}|W|^2W(t - t_0)\mathrm{e}^{i\omega t} + \frac{W^3}{8b^2\omega^2}\{(3\omega^2 - b^2) - 4ib\omega\}\mathrm{e}^{3i\omega t}\Big] + \text{c.c.} \\ \eta_2 \;=\; & \frac{|W|^2W}{6b^3}\Big[-(b^2 + \omega^2)(t - t_0) + \frac{1}{\omega}\{2b\omega + i(b^2 - \omega^2)\}\Big]\mathrm{e}^{i\omega t} \\ & + \frac{W^3}{8b^3}\{-4b + \frac{i}{\omega}(3b^2 - \omega^2)\}\mathrm{e}^{3i\omega t} + \text{c.c.}. \end{aligned} \tag{5.42}$$

The RG/E equation reads

$$\frac{d\mathbf{u}}{dt_0} = \mathbf{0}, \tag{5.43}$$

with $t_0 = t$, which gives the equation for $W(t)$ as

$$\frac{dW}{dt} = -i\epsilon^2 \frac{\omega^2 + b^2}{6\omega b^2}|W|^2 W. \tag{5.44}$$

If we define $A(t)$ and $\theta(t)$ by $W(t) = (A(t)/2i)\exp i\theta(t)$, the equation gives

$$A(t) = \text{const.}, \qquad \theta(t) = -\frac{\epsilon^2 A^2}{24}(1 + \frac{b^2}{\omega^2})\omega t + \bar{\theta}_0, \tag{5.45}$$

with $\bar{\theta}_0$ being a constant. Owing to the prefactor i in r.h.s. of Eq. (5.44), the absolute value of the amplitude A becomes independent of t, while the phase θ has a t-dependence. The envelope function is given by

$$\mathbf{u}_E(t) = \begin{pmatrix} \xi_E(t) \\ \eta_E(t) \end{pmatrix} = \mathbf{u}(t, t_0)\Big|_{t_0 = t, \partial\mathbf{u}/\partial t_0 = 0}. \tag{5.46}$$

In terms of the components, one has

$$\begin{aligned}
\xi_{E} &= A\sin\Theta(t) - \epsilon\frac{A^2}{6\omega}(\sin 2\Theta(t) + \frac{2\omega}{b}\cos 2\Theta(t)) \\
&\quad -\frac{\epsilon^2 A^3}{32}\frac{3\omega^2 - b^2}{\omega^2 b^2}(\sin 3\Theta(t) - \frac{4\omega b}{3\omega^2 - b^2}\cos 3\Theta(t)), \\
\eta_{E} &= -\frac{\omega}{b}\Bigg[\Big(A - \frac{\epsilon^2 A^3}{24}\frac{b^2 - \omega^2}{b^2\omega^2}\Big)\cos\Theta(t) - \frac{\epsilon^2 A^3}{12 b\omega}\sin\Theta(t) \\
&\quad +\epsilon\frac{A^2}{2b}\Big(\sin 2\Theta(t) - \frac{2b}{3\omega}\cos 2\Theta(t)\Big) - \frac{\epsilon^2 A^3}{8b\omega}\Big(\sin 3\Theta(t) - \frac{3b^2 - \omega^2}{4b^2\omega^2}\cos 3\Theta(t)\Big)\Bigg]
\end{aligned} \tag{5.47}$$

where

$$\Theta(t) \equiv \tilde{\omega}t + \bar{\theta}_0, \qquad \tilde{\omega} \equiv \{1 - \frac{\epsilon^2 A^2}{24}(1 + \frac{b^2}{\omega^2})\}\omega. \tag{5.48}$$

One sees that the angular frequency is shifted.

We identify $\mathbf{u}_E(t) = (\xi_E(t), \eta_E(t))$ as an approximate solution to Eq.(5.28). According to the basic theorem presented in section 4, $\mathbf{u}_E(t)$ is an approximate but uniformly valid solution to the equation up to $O(\epsilon^3)$. We remark that the resultant trajectory is closed in conformity with the conservation law given in Eq. (5.26).

"Explicit solutions" of two-pieces of Lotka-Volterra equation were considered by Frame [14]; however, his main conceren was on extracting the period of the solutions in an average method. Comparing the Frame's method, the RG method is simpler, more transparent and explicit. The present author is not aware of any other work which gives an explicit form of the solution as given in Eq. (5.47,48).

5.3 The Lorenz model

The Lorenz model[12] for the thermal convection is given by

$$\begin{aligned} \dot{\xi} &= \sigma(-\xi + \eta), \\ \dot{\eta} &= r\xi - \eta - \xi\zeta, \\ \dot{\zeta} &= \xi\eta - b\zeta. \end{aligned} \tag{5.49}$$

The steady states are give by

$$\text{(A)} \;\; (\xi, \eta, \zeta) = (0, 0, 0), \quad \text{(B)} \;\; (\xi, \eta, \zeta) = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1). \tag{5.50}$$

The linear stability analysis[10] shows that the origin is stable for $0 < r < 1$ but unstable for $r > 1$, while the latter steady states (B) are stable for $1 < r < \sigma(\sigma + b + 3)/(\sigma - b - 1) \equiv r_c$ but unstable for $r > r_c$. In this paper, we examine the non-linear stability around the origin for $r \sim 1$; we put

$$r = 1 + \mu \quad \text{and} \quad \mu = \chi\epsilon^2, \quad \chi = \text{sgn}\mu. \tag{5.51}$$

We expand the quantities as Taylor series of ϵ:

$$\mathbf{u} \equiv \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \epsilon\mathbf{u}_1 + \epsilon^2\mathbf{u}_2 + \epsilon^3\mathbf{u}_3 + \cdots, \tag{5.52}$$

where $\mathbf{u}_i = {}^t(\xi_i, \eta_i, \zeta_i)$ $(i = 1, 2, 3, \ldots)$. The first order equation reads

$$\left(\frac{d}{dt} - L_0\right)\mathbf{u}_1 = \mathbf{0}, \tag{5.53}$$

where

$$L_0 = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}, \tag{5.54}$$

the eigenvalues of which are found to be

$$\lambda_1 = 0, \quad \lambda_2 = -\sigma - 1, \quad \lambda_3 = -b. \tag{5.55}$$

The respective eigenvectors are

$$\mathbf{U}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{U}_2 = \begin{pmatrix} \sigma \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{U}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{5.56}$$

When we are interested in the asymptotic state as $t \to \infty$, one may take the neutrally stable solution

$$\mathbf{u}_1(t; t_0) = W(t_0)\mathbf{U}_1, \tag{5.57}$$

where we have made it explicit that the solution may depend on the initial time t_0, which is supposed to be close to t. In terms of the components,

$$\xi_1(t) = W(t_0), \quad \eta_1(t) = W(t_0), \quad \zeta_1(t) = 0. \tag{5.58}$$

The second order equation now reads

$$\Big(\frac{d}{dt} - L_0\Big)\mathbf{u}_2 = \begin{pmatrix} 0 \\ -\xi_1\zeta_1 \\ \xi_1\eta_1 \end{pmatrix} = W^2\mathbf{U}_3, \tag{5.59}$$

which yields

$$\mathbf{u}_2(t) = \frac{W^2}{b}\mathbf{U}_3, \tag{5.60}$$

or in terms of the components

$$\xi_2 = \eta_2 = 0, \quad \zeta_2 = \frac{W^2}{b}. \tag{5.61}$$

Then the third order equation is given by

$$\Big(\frac{d}{dt} - L_0\Big)\mathbf{u}_3 = \begin{pmatrix} 0 \\ -\chi\xi_1 - \xi_2\zeta_1 - \xi_1\zeta_2 \\ \xi_2\eta_1 + \xi_1\eta_2 \end{pmatrix} = \frac{1}{1+\sigma}(\chi W - \frac{1}{b}W^3)(\sigma\mathbf{U}_1 - \mathbf{U}_2), \tag{5.62}$$

which yields

$$\mathbf{u}_3 = \frac{1}{1+\sigma}(\chi W - \frac{1}{b}W^3)\{\sigma(t - t_0 + \frac{1}{1+\sigma})\mathbf{U}_1 - \frac{1}{1+\sigma}\mathbf{U}_2\}. \tag{5.63}$$

Thus gathering all the terms, one has

$$\begin{aligned}\mathbf{u}(t;t_0) &= \epsilon W(t_0)\mathbf{U}_1 + \frac{\epsilon^2}{b}W(t_0)^2\mathbf{U}_3 \\ &\quad + \frac{\epsilon^3}{1+\sigma}(\chi W(t_0) - \frac{1}{b}W(t_0)^3)\{\sigma(t - t_0 + \frac{1}{1+\sigma})\mathbf{U}_1 - \frac{1}{1+\sigma}\mathbf{U}_2\},\end{aligned} \tag{5.64}$$

up to $O(\epsilon^4)$. The RG/E equation now reads

$$\begin{aligned}\mathbf{0} &= \frac{d\mathbf{u}}{dt_0}\Big|_{t_0=t}, \\ &= \epsilon\frac{dW}{dt}\mathbf{U}_1 + 2\frac{\epsilon^2}{b}W\frac{dW}{dt}\mathbf{U}_3 - \frac{\sigma}{1+\sigma}\epsilon^3(\chi W - \frac{1}{b}W^3)\mathbf{U}_1,\end{aligned} \tag{5.65}$$

up to $O(\epsilon^4)$. Noting that one may self-consistently assume that $dW/dt = O(\epsilon^2)$, we have the amplitude equation

$$\frac{dW}{dt} = \epsilon^2\frac{\sigma}{1+\sigma}(\chi W(t) - \frac{1}{b}W(t)^3). \tag{5.66}$$

With this $W(t)$, the envelope function is given by

$$\begin{aligned}\mathbf{u}_E(t) &= \mathbf{u}(t;t_0=t), \\ &= \epsilon W(t)\mathbf{U}_1 + \frac{\epsilon^2}{b}W(t)^2\mathbf{U}_3 + \frac{\epsilon^3}{(1+\sigma)^2}(\chi W(t) - \frac{1}{b}W(t)^3)(\sigma\mathbf{U}_1 - \mathbf{U}_2),\end{aligned} \tag{5.67}$$

or

$$\begin{aligned}\xi_E(t) &= \epsilon W(t), \\ \eta_E(t) &= \epsilon W(t) + \frac{\epsilon^3}{1+\sigma}(\chi W(t) - \frac{1}{b}W(t)^3), \\ \zeta_E(t) &= \frac{\epsilon^2}{b}W(t)^2.\end{aligned} \tag{5.68}$$

We may identify the envelope functions thus constructed as a global solution to the Lorenz model; according to the general theorem given in section 4, the envelope functions satisfy Eq.(5.49) approximately but uniformly for $\forall t$ up to $O(\epsilon^4)$.

A remark is in order here; Eq.(5.68) shows that the slow manifold which may be identified with a center manifold[10] is given by

$$\eta = (1+\epsilon^2\frac{\chi}{1+\sigma})\xi - \frac{1}{b(1+\sigma)}\xi^3, \quad \zeta = \frac{1}{b}\xi^2. \tag{5.69}$$

Notice here that the RG method is also a powefull tool to extract center manifolds in a concrete form. It is worth mentioning that since the RG method utilizes neutrally stable solutions as the unperturbed ones, it is rather natural that the RG method can extract center manifolds when exist. The applicability of the RG method was discussed in [4] using a generic model having a center manifold, although the relation between the exitence of center manifolds and neutrally stable solutions is not so transparent in their general approach.

6 Bifurcation Theory

In this section, we take generic equations with a bifurcation. We shall derive the Landau-Stuart and Ginzburg-Landau equations in the RG method. In this section, we shall follow Kuramoto's monograph[8] for notations to clarify the correspondence between the RG method and the reductive perturbation (RP) method.

6.1 Landau-Stuart equation

We start with the n-dimensional equation

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}, t; \mu). \tag{6.1}$$

Let $\mathbf{X}_0(\mu)$ is a steady solution

$$\mathbf{F}(\mathbf{X}_0(\mu);\mu) = 0. \tag{6.2}$$

Shifting the variable as $\mathbf{X} = \mathbf{X}_0 + \mathbf{u}$, we have a Taylor series

$$\frac{d\mathbf{u}}{dt} = L\mathbf{u} + M\mathbf{u}\mathbf{u} + N\mathbf{u}\mathbf{u}\mathbf{u} + \cdots, \tag{6.3}$$

where we have used the diadic and triadic notations[8];

$$\begin{aligned} L_{ij} &= \left.\frac{\partial F_i}{\partial X_j}\right|_{\mathbf{X}=\mathbf{X}_0}, \quad (M\mathbf{u}\mathbf{u})_i = \sum_{j,k} \frac{1}{2} \left.\frac{\partial^2 F_i}{\partial X_j \partial X_k}\right|_{\mathbf{X}=\mathbf{X}_0} u_j u_k, \\ (N\mathbf{u}\mathbf{u}\mathbf{u})_i &= \sum_{j,k,l} \frac{1}{6} \left.\frac{\partial^3 F_i}{\partial X_j \partial X_k \partial X_l}\right|_{\mathbf{X}=\mathbf{X}_0} u_j u_k u_l. \end{aligned} \tag{6.4}$$

We suppose that when $\mu < 0$, $\mathbf{X}_0$ is stable for sufficiently small perturbations, while when $\mu > 0$, otherwise. We also confine ourselves to the case where a Hopf bifurcation occurs. We expand L, M and N as

$$L = L_0 + \mu L_1 + \cdots, \quad M = M_0 + \mu M_1 + \cdots, \quad N = N_0 + \mu N_1 + \cdots. \tag{6.5}$$

The eigenvalues λ^α $(\alpha = 1, 2, \ldots, n)$ of L are also expanded as

$$\lambda^\alpha = \lambda_0^\alpha + \mu \lambda_1^\alpha + \cdots, \tag{6.6}$$

with

$$L_0 \mathbf{U}_\alpha = \lambda_0^\alpha \mathbf{U}_\alpha. \tag{6.7}$$

We assume that $\lambda_0^1 = -\lambda_0^2$ are pure imaginary, i.e., $\lambda_0^1 = i\omega_0$, and $\mathrm{Re}\lambda_0^\alpha < 0$ for $\alpha = 3, 4, \ldots$.

Defining ϵ and χ by $\epsilon = \sqrt{|\mu|}$ and $\chi = \mathrm{sgn}\mu$, we expand as

$$\mathbf{u} = \epsilon \mathbf{u}_1 + \epsilon^2 \mathbf{u}_2 + \epsilon^3 \mathbf{u}_3 + \cdots. \tag{6.8}$$

The $\mathbf{u}_i$ $(i = 1, 2, 3, \ldots)$ satisfies

$$\begin{aligned} \left(\frac{d}{dt} - L_0\right)\mathbf{u}_1 &= \mathbf{0}, \\ \left(\frac{d}{dt} - L_0\right)\mathbf{u}_2 &= M_0 \mathbf{u}_1 \mathbf{u}_1, \\ \left(\frac{d}{dt} - L_0\right)\mathbf{u}_3 &= \chi L_1 \mathbf{u}_1 + 2M_0 \mathbf{u}_1 \mathbf{u}_2 + N_0 \mathbf{u}_1 \mathbf{u}_1 \mathbf{u}_1, \end{aligned} \tag{6.9}$$

etc.

To see the asymptotic behavior as $t \to \infty$, we take the neutrally stable solution as the lowest one around $t = t_0$;

$$\mathbf{u}_1(t;t_0) = W(t_0)\mathbf{U}\mathrm{e}^{i\omega_0 t} + \text{c.c.}, \tag{6.10}$$

where c.c. stands for the complex conjugate. With this choice, we have only two degrees of freedom for the initial value $W(t_0)$.

The second order equation is solved easily to yield

$$\mathbf{u}_2(t;t_0) = (\mathbf{V}_+ W(t_0)^2 \mathrm{e}^{2i\omega_0 t} + \text{c.c.}) + \mathbf{V}_0 |W(t_0)|^2, \tag{6.11}$$

where

$$\mathbf{V}_+ = -(L_0 - 2i\omega_0)^{-1} M_0 \mathbf{U}\mathbf{U}, \quad \mathbf{V}_0 = -2L_0^{-1} M_0 \mathbf{U}\bar{\mathbf{U}}, \tag{6.12}$$

with $\bar{\mathbf{U}}$ being the complex conjugate of $\mathbf{U}$.[7] Inserting $\mathbf{u}_1$ and $\mathbf{u}_2$ into the r.h.s of Eq. (6.9), we get

$$\begin{aligned} \left(\frac{d}{dt} - L_0\right)\mathbf{u}_3 &= \{\chi L_1 W\mathbf{U} + (2M_0\bar{\mathbf{U}}\mathbf{V}_+ + 3N_0\mathbf{U}\mathbf{U}\bar{\mathbf{U}})|W|^2 W\}\mathrm{e}^{i\omega_0 t} + \text{c.c.} + \text{h.h.}, \\ &\equiv \mathbf{A}\mathrm{e}^{i\omega_0 t} + \text{c.c.} + \text{h.h.}, \end{aligned} \tag{6.13}$$

where h.h. stands for higher harmonics. So far, the discussion is a simple perturbation theory and has proceeded in the same way as given in the RP method except for not having introduced multiple times.

Now we expand $\mathbf{A}$ by the eigenvectors $\mathbf{U}_\alpha$ of L_0 as

$$\mathbf{A} = \sum_\alpha A_\alpha \mathbf{U}_\alpha, \tag{6.14}$$

where

$$A_\alpha = \mathbf{U}_\alpha^* \mathbf{A}. \tag{6.15}$$

Here $\mathbf{U}_\alpha^*$ satisfies

$$\mathbf{U}_\alpha^* L_0 = \lambda_0^\alpha L_0, \tag{6.16}$$

and is normalized as $\mathbf{U}_\alpha^* \mathbf{U}_\alpha = 1$.

Then we get for $\mathbf{u}_3$

$$\mathbf{u}_3(t;t_0) = \{A_1(t - t_0 + \delta)\mathbf{U} + \sum_{\alpha \neq 1} \frac{A_\alpha}{i\omega_0 - \lambda_0^\alpha}\mathbf{U}_\alpha\}\mathrm{e}^{i\omega_0 t} + \text{c.c.} + \text{h.h.}. \tag{6.17}$$

[7] In other sections, we use the notation a^* for the complex conjugate of a. In this section, * is used for a different meaning, following ref.[8]; see Eq. (6.16).

The constant δ is chosen so that the coefficient of the secular term of the first component vanishes at $t = t_0$. Note the appearance of the secular term which was to be avoided in the RP method: The condition for the secular terms to vanish is called the solvability condition which plays the central role in the RP method[8].

Thus we finally get

$$\mathbf{u}(t;t_0) = \{\epsilon W(t_0)\mathbf{U} + \epsilon^3(A_1(t - t_0 + \delta)\mathbf{U} + \sum_{\alpha\neq 1} \frac{A_\alpha}{i\omega_0 - \lambda_0^\alpha}\mathbf{U}_\alpha)\}\mathrm{e}^{i\omega_0 t} + \text{c.c.} + \text{h.h.}. \quad (6.18)$$

The RG/E equation

$$\left.\frac{d\mathbf{u}}{dt_0}\right|_{t_0=t} = \mathbf{0}, \quad (6.19)$$

yields

$$\begin{aligned} \frac{dW}{dt} &= \epsilon^2 A_1, \\ &= \epsilon^2[\chi \mathbf{U}^* L_1 \mathbf{U} W + \{2\mathbf{U}^* M_0 \bar{\mathbf{U}}\mathbf{V}_+ + 3\mathbf{U}^* N_0 \mathbf{U}\mathbf{U}\bar{\mathbf{U}}\}|W|^2 W], \end{aligned} \quad (6.20)$$

up to $O(\epsilon^3)$. Here note that the terms coming from h.h. do not contribute to this order because dW/dt_0 is $O(\epsilon^2)$. The resultant equation is so called the Landau-Stuart equation and coincides with the result derived in the RP method[8].

6.2 The Ginzburg-Landau equation

We add the diffusion term to Eq.(6.1);

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}) + D\nabla^2\mathbf{X}, \quad (6.21)$$

where D is a diagonal matrix. Let $\mathbf{X}_0$ be a uniform and steady solution.

Shifting the variable $\mathbf{X} = \mathbf{X}_0 + \mathbf{u}$ as before, we have

$$\frac{d\mathbf{u}}{dt} = \hat{L}\mathbf{u} + M\mathbf{u}\mathbf{u} + N\mathbf{u}\mathbf{u}\mathbf{u} + \cdots, \quad (6.22)$$

with

$$\hat{L} = L + D\nabla^2. \quad (6.23)$$

Then using the same expansion as before, we have the same equation for $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{u}_3$ as given in Eq.(6.9) with L_0 being replaced with $\hat{L}_0 \equiv L_0 + D\nabla^2$.

To see the asymptotic behavior as $t \to \infty$, we take the neutrally stable uniform solution as the lowest one around $t = t_0$ and $\mathbf{r} = \mathbf{r}_0$;

$$\mathbf{u}_1(t, \mathbf{r}; t_0, \mathbf{r}_0) = W(t_0, \mathbf{r}_0)\mathbf{U}e^{i\omega_0 t} + \text{c.c.}. \tag{6.24}$$

With this choice, we have only two degrees of freedom for the initial value $W(t_0, \mathbf{r}_0)$.

The second order equation is solved easily to yield the same form as that given in Eq.(6.11). Inserting $\mathbf{u}_1$ and $\mathbf{u}_2$ into the r.h.s of Eq. (6.9) with L_0 replaced with $\hat{L}_0$, we have

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \hat{L}_0\Big)\mathbf{u}_3 &= \{\chi L_1 W\mathbf{U} + (2M_0\bar{\mathbf{U}}\mathbf{V}_+ + 3N_0\mathbf{U}\mathbf{U}\bar{\mathbf{U}})|W|^2W\}e^{i\omega_0 t} + \text{c.c.} + \text{h.h.}, \\ &\equiv \mathbf{A}e^{i\omega_0 t} + \text{c.c.} + \text{h.h.}. \end{aligned} \tag{6.25}$$

Then we get for $\mathbf{u}_3$ in the spatially 1-dimensional case,

$$\begin{aligned}\mathbf{u}_3(t; t_0) &= \Big[A_1\{c_1(t - t_0 + \delta) - \frac{c_2}{2}D^{-1}(x^2 - x_0^2 + \delta')\}\mathbf{U} + \sum_{\alpha\neq 1}\frac{A_\alpha}{i\omega_0 - \lambda_0^\alpha}\mathbf{U}_\alpha\Big]e^{i\omega_0 t} \\ &\quad + \text{c.c.} + \text{h.h.}, \end{aligned} \tag{6.26}$$

with $c_1 + c_2 = 1$. We have introduced constants δ and δ' so that the secular terms of the first component of $\mathbf{u}_3$ vanish at $t = t_0$ and $x = x_0$. Note the appearance of the secular terms both t- and x-directions; these terms were to be avoided in the RP method with the use of the solvability condition.

Adding all the terms, we finally get

$$\begin{aligned}\mathbf{u}(t; t_0) &= \Big[(\epsilon W(t_0, x_0)\mathbf{U} + \epsilon^3\{A_1\Big(c_1(t - t_0 + \delta) - \frac{c_2}{2}D^{-1}(x^2 - x_0^2 + \delta')\Big)\mathbf{U} \\ &\quad + \sum_{\alpha\neq 1}\frac{A_\alpha}{i\omega_0 - \lambda_0^\alpha}\mathbf{U}_\alpha\}\Big]e^{i\omega_0 t} + \text{c.c.} + \text{h.h.}, \end{aligned} \tag{6.27}$$

up to $O(\epsilon^4)$. The RG/E equation[8]

$$\frac{d\mathbf{u}}{dt_0}\Big|_{t_0 = t} = \mathbf{0}, \quad \frac{d\mathbf{u}}{dx_0}\Big|_{x_0 = x} = \mathbf{0}, \tag{6.28}$$

yields

$$\frac{\partial W}{\partial t} = \epsilon^2 c_1 A_1 + O(\epsilon^3), \qquad D\frac{\partial W}{\partial x} = -\epsilon^2 x c_2 A_1 + O(\epsilon^3). \tag{6.29}$$

We remark that the seemingly vector equation is reduced to a scalar one. Differentiating the second equation once again, we have

$$D\frac{\partial^2 W}{\partial x^2} = -\epsilon^2 c_2 A_1 + O(\epsilon^3). \tag{6.30}$$

[8]See section 3.

Here we have utilized the fact that $\partial W/\partial x = O(\epsilon^2)$. Noting that $c_1 + c_2 = 1$, we finally reach

$$\begin{aligned}\frac{\partial W}{\partial t} - D\frac{\partial^2 W}{\partial x^2} &= \epsilon^2 A_1, \\ &= \epsilon^2[\chi \mathbf{U}^* L_1 \mathbf{U} W + \{2\mathbf{U}^* M_0 \bar{\mathbf{U}} \mathbf{V}_+ + 3\mathbf{U}^* N_0 \mathbf{U}\mathbf{U}\bar{\mathbf{U}}\}|W|^2 W], \quad (6.31)\end{aligned}$$

up to $O(\epsilon^3)$. This is so called the time-dependent Ginzburg-Landau (TDGL) equation and coincides with the amplitude equation derived in the RP method[8].

We have seen that the RG method can reduce the dynamics of a class of non-linear equations as the RP method can. Therefore it is needless to say that our method can be applied to the Brusselators[15], for instance, and leads to the same amplitude equations as the RP method[8] does[16].

7 A brief summary and concluding remarks

In this paper, we have shown that the RG method of Goldenfeld, Oono and their collaborators can be equally applied to vector equations, i.e., systems of ODE's and PDE's, as to scalar equations.[3, 4, 5, 6] We have formulated the method on the basis of the classical thoery of envelopes, thereby completed the argument given in [5, 6]. We have worked out for some examples of systems of ODE's, i.e., the forced Duffing, the Lotka-Volterra and the Lorenz equation. It has been also shown in a generic way that the method applied to equations with a bifurcation leads to the amplitude equations, such as the Landau-Stuart and the (time-dependent) Ginzburg-Landau equation.

Then how about the phase equations[8]?: The phase equations describe another reduced dynamics. The basis of the reduction of the dynamics by the phase equations lies in the fact that when a symmetry is broken, there appears a slow motion which is a classical counter part of the Nambu-Goldstone boson in quantum field theory. We believe that if the phase equations are related to slow motions of the system at all, the RG method should also leads to the phase equations. It is an interesting task to be done to show that it is the case.

There is another class of dynamics than those described by differential equations, i.e., difference equations or discrete maps. It is interesting that a natural extension of the RG/E equation to difference equations leads to a reduction of the dynamics.[9] This fact suggests that the RG method pioneered by Goldenfeld , Oono and their collaborators provides one of the most promising candidate for a general theory of the reduction of dynamics, although it is certain that such a mechanical and general method is often tedious in the actual calculations.[9] As an application of the reduction of difference equations, it

[9] It should be mentioned that there are other methods [17, 18] for the dynamical reduction as promising as the RG and RP method are.

will be interesting to see whether the coupled map lattice equations as systems of non-linear difference equations[19] can be reduced to simpler equations by the RG method. We hope that we can report about it in the near future.

Acknowledgements

This work is partly a reply to some people who asked if the RG method could be applied to vector equations. The author acknowledges M. Davis and Y. Kuramoto for questions and comments for the previous papers[5, 6]. He also thanks J. Matsukidaira and T. Ikeda for indicating the significance of examining vector equations. J Matsukidaira is gratefully acknowledged for useful comments in the earlist stage of this work. He thanks M. Yamaguti and Y. Yamagishi for discussions on difference equations. He is indebted to R. Hirota, H. Matano, Y. Nishiura, J. Satsuma and M. Yamaguti for their interest in this work. He expresses his sincere thanks to M. Yamaguti for his encouragement.

Note added After submitting the paper, the author was informed that S. Sasa applied the RG method to derive phase equations in a formal way. The author is grateful to S. Sasa for sending me the TEX file(`patt-sol/9608008`) of the paper before its publication.

Appendix A

In this Appendix, we give a quick review of Goldenfeld, Oono and their collaborators' prescription for the RG method. Then we summarize the problems to which a mthematical reasoning is needed in the author's point of view.

We take the following simplest example to show their prescription:

$$\frac{d^2x}{dt^2} \ + \ \epsilon\frac{dx}{dt} \ + \ x \ = \ 0, \tag{A.1}$$

where ϵ is supposed to be small. The exact solution reads

$$x(t) = A\exp(-\frac{\epsilon}{2}t)\sin(\sqrt{1-\frac{\epsilon^2}{4}}t+\theta), \tag{A.2}$$

where A and θ are constant to be determined by an initial condition.

Now, let us blindly apply the perturbation theory expanding x as

$$x(t) = x_0(t) \ + \ \epsilon x_1(t) \ + \ \epsilon^2 x_2(t) \ + \ .... \tag{A.3}$$

The result is found to be[5]

$$\begin{aligned} x(t;t_0) \ &= \ A_0\sin(t+\theta_0) - \epsilon\frac{A_0}{2}(t-t_0)\sin(t+\theta_0) \\ &+\epsilon^2\frac{A_0}{8}\{(t-t_0)^2\sin(t+\theta_0) - (t-t_0)\cos(t+\theta_0)\} + O(\epsilon^3). \end{aligned} \tag{A.4}$$

Now here come the crucial steps of the Goldenfeld et al's prescription:

(i) First they introduce a dummy time τ which is close to t, and "renormalize" $x(t;t_0)$ by writing $t-t_0 = t-\tau+\tau-t_0$;

$$\begin{aligned} x(t,\tau) \ &= \ A(\tau)\sin(t+\theta(\tau)) - \epsilon\frac{A(\tau)}{2}(t-\tau)\sin(t+\theta(\tau)) \\ &+\epsilon^2\frac{A(\tau)}{8}\{(t-\tau)^2\sin(t+\theta(\tau)) - (t-\tau)\cos(t+\theta(\tau))\} + O(\epsilon^3), \end{aligned} \tag{A.5}$$

with

$$x(\tau,\tau) = A(\tau)\sin(\tau+\theta(\tau)). \tag{A.6}$$

Here A_0 and θ_0 have been multiplicatively renormalized to $A(\tau)$ and $\theta(\tau)$.

(ii) They observe that τ is an arbitrary constant introduced by hand, thus they claim that the solution $x(t,\tau)$ should not depend on τ; namely, $x(t,\tau)$ should satisfy the equation

$$\frac{dx(t,\tau)}{d\tau} = 0. \tag{A.7}$$

This is similar to the RG equation in the field theory where τ corresponds to the renormalization point τ; hence the name of the RG method.

(iii) Finally they impose another important but a mysterious condition that

$$\tau = t. \tag{A.8}$$

From (ii) and (iii), one has

$$\frac{dA}{d\tau} + \frac{\epsilon}{2}A = 0, \quad \frac{d\theta}{d\tau} + \frac{\epsilon^2}{8} = 0, \tag{A.9}$$

which gives

$$A(\tau) = \bar{A}\mathrm{e}^{-\epsilon\tau/2}, \quad \theta(\tau) = -\frac{\epsilon^2}{8}\tau + \bar{\theta}, \tag{A.10}$$

where $\bar{A}$ and $\bar{\theta}$ are constant numbers. Thus, rewriting τ to t in $x(\tau)$, one gets

$$x(t,t) = \bar{A}\exp(-\frac{\epsilon}{2}t)\sin((1-\frac{\epsilon^2}{8})t + \bar{\theta}). \tag{A.11}$$

They identify $x(t,t)$ with the desired solution $x(t)$. Then one finds that the resultant $x(t)$ is an approximate but uniformly valid solution to Eq.(A.1). In short, the solution obtained in the perturbation theory with the local nature has been "improved" by the RG equation Eq.(A.7) to become a global solution.

But what have we done mathematically? what is a mathematical meaning of the "renormalization" replacing t_0 with the extra dummy time τ? Can't we avoid the "renormalization" procedure to solve a purely mathematical problem? Why can we identify $x(t,t)$ with the desired solution?; with τ being a constant, $x(t,\tau)$ can be a(n) (approximate) solution to Eq. (A.1), can't it? In other words, when the operator d/dt hits the second argument of $x(t,t)$, what happens?

In Ref.[5], it was shown that the "renormalization" procedure to introduce the extra dummy time τ is not necessary. Furthermore, it was clarified that the conditions (ii) and (iii) are the ones to construct the *envelope* of the family of the local solutions obtained in the perturbation theory; $x(t;t)$ is the envelope function of the family of curves given by $x(t;t_0)$ where t_0 parametrizes the curves in the family. Furthermore, it was shown that the envelope function $x(t,t)$ satisfies the orginal equations approximately but uniformly; the hitting of d/dt on the second argument of $x(t,t)$ does not harm anything. In short, the prescription given by Goldenfeld, Oono and their collaborators is not incorrect, but the reasoning for the prescription is given in [5, 6] and will be more refined in the present paper. In Ref.[6], a simplification of the prescription and its mathematical foundation is given for PDE's.

Appendix B

In this Appendix, we solve the forced Duffing equation without converting it to a system. It is easier to solve it in this way than in the way shown in the text.

We start with Eq. (2.6)

$$\ddot{z} + 2\epsilon\gamma\dot{z} + (1+\epsilon\sigma)z + \frac{\epsilon h}{2}(3|z|^2 z + z^{*3}) = \epsilon f \mathrm{e}^{it}, \tag{B.1}$$

where ϵ is small.

Expanding z as

$$z = z_0 + \epsilon z_1 + \epsilon^2 z_2 + \cdots, \tag{B.2}$$

one gets for z in the perturbation theory

$$z(t;t_0) = W(t_0)\mathrm{e}^{it} + \epsilon(t-t_0)\{f - W(\sigma+2i\gamma) - \frac{3h}{2}|W|^2 W\}\mathrm{e}^{it} + \epsilon\frac{1}{16}W^*(t)^3\mathrm{e}^{3it} + O(\epsilon^2) \tag{B.3}$$

Note that there exists a secular term in the first order term.

The RG/E equation reads[5]

$$\frac{dz}{dt_0} = 0 \tag{B.4}$$

with $t_0 = t$, which leads to

$$\dot{W} = -\epsilon(\sigma+2i\gamma)W - \frac{3}{2}\epsilon h|W|^2 W + \epsilon f \tag{B.5}$$

up to $O(\epsilon^2)$. Here we have discarded terms such as $\epsilon dW/dt$, which is $O(\epsilon^2)$ because $dW/dt = O(\epsilon)$. The resultant equation for the amplitude is the Landau-Stuart equation for the Duffing equation. The envelope is given

$$z_E(t) = z(t;t_0=t) = W(t)\mathrm{e}^{it} + \frac{\epsilon}{16}W^{*3}\mathrm{e}^{3it} + O(\epsilon^2). \tag{B.6}$$

We identify $z_E(t)$ with a global solution of Eq.(B.2), and $x(t) = \mathrm{Re}[z_E]$ and $y(t) = \mathrm{Im}[z_E]$ are solutions to Eq.(B.1). As shown in the text, $\forall t$, $z_E(t)$ satisfies Eq.(B.2) uniformly up to $O(\epsilon^2)$.

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